The Kac-Ward approach to the Ising Model

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The critical point and the spectral radius of the transition matrix

In this chapter, we continue in the spirit of Section 2.3, where the spectral radius and operator norm of the Kac–Ward transition matrices were considered. We explicitly compute the operator norm of what we call the conjugated transition matrix defined for a general graph in the plane, and hence we provide an upper bound on the spectral radius of the standard Kac–Ward transition matrix. Combining this result with the Kac–Ward formula for the high- and low-temperature expansion of the partition function yields domains of parameters of the model where there is no phase transition. We will focus only on the analytic properties of the free energy, but our bounds, together with the methods from Chapters 2 and 3 allow also to identify regions where there is spontaneous magnetization or exponential decay of the correlation functions. The advantage of our approach is that it does not require any form of periodicity of the underlying graph.

Moreover, our results are optimal for the Ising model defined on isoradial graphs with uniformly bounded rhombus angles (see condition (4.5)), i.e. we can conclude that the self-dual Z-invariant coupling constants, first considered by Baxter [4], are indeed critical in the classical sense. To be more precise, after introducing the inverse temperature parameter β to the corresponding Ising model, we show that the thermodynamic limits of the free energy density can have singularities only at $\beta = 1$. The isoradial graphs, or equivalently rhombic lattices, were introduced by Duffin [23] as potentially the largest family of graphs where one can do discrete complex analysis. As mentioned in [14], this class of graphs seems to be the most general family of graphs where the critical Ising model can be defined in terms of the local geometry of the graph, and it also seems to be the one, where our bounds for the spectral radius and operator norm of the Kac–Ward transition matrix yield the critical point of the Ising model.

The self-dual Z-invariant Ising model has been extensively studied in the mathematics literature. Chelkak and Smirnov [14] proved that the associated discrete holomorphic fermion has a universal, conformally invariant scaling limit. Boutillier and de Tilière [8, 9] gave a complete description of the corresponding dimer model, yielding also an alternative proof of Baxter's formula for the critical free energy density. Mercat [46] defined a notion of criticality for discrete Riemann surfaces and investigated its connection with criticality in the Ising model. The self-dual Z-invariant Ising model is commonly referred to as critical. However, criticality in the statistical mechanics sense has been established only in the case of doubly periodic isoradial graphs (see Example 1.6 of [18] and the references therein). As already mentioned, we extend this result to a wide class of aperiodic isoradial graphs.

This chapter is organized as follows: in Section 4.1, we describe the graphs that we will work with, we revisit the definition of the Ising model and the notion of phase transition, and we state our main theorem. Section 4.2 defines the Kac–Ward operator and presents its connection to the Ising model. It also contains our results for the Kac–Ward transition matrix. The proof of the main theorem is postponed until Section 4.3.

4.1 Results for the Ising model

Let $\Gamma = (V_\Gamma, E_\Gamma)$ be an infinite, simple, planar graph embedded in the complex plane and let $\Gamma^* = (V^*_\Gamma, E^*_\Gamma)$ be its planar dual. We assume that both Γ and Γ^* have uniformly bounded vertex degrees. One should think of Γ as any kind of tiling or discretization of the plane. In particular, Γ can be a regular lattice, or an instance of an isoradial graph (see Section 4.1.2). We call a subgraph $\mathcal{G} = (V, E)$ of Γ a *subtiling* if there is a collection of faces of Γ, such that $\mathcal{G}$ is the subgraph induced by all the edges forming boundaries of these faces. We define the boundary $\partial\mathcal{G}$ of a subtitling $\mathcal{G}$ to be the set of vertices of $\mathcal{G}$ which lie on the boundary of at least one face which is not in the defining collection of faces. Note that for subtilings containing holes, i.e. where the union of the closed faces is not simply connected, this definition is not the same as the one from Chapter 3, where only the outermost boundary vertices were included. The reason for that is that in Chapter 3, we did not consider any reference graph Γ. As before, we identify vertices of graphs embedded in the plane with the corresponding complex numbers.

Let $J = (J_e)_{e\in E_\Gamma}$ be a vector of ferromagnetic coupling constants. For each finite subtiling $\mathcal{G}$, we will consider an Ising model on $\mathcal{G}$ defined by J and the inverse temperature parameter β. Recall that

$$\Omega^{\text{free}}_{\mathcal{G}} = \{-1,+1\}^V \quad \text{and} \quad \Omega^{+}_{\mathcal{G}} = \{\sigma \in \Omega^{\text{free}}_{\mathcal{G}} : \sigma_v = +1 \text{ if } v \in \partial\mathcal{G}\}$$

are the spaces of spin configurations with free and positive boundary conditions. The Ising model with $\square$ boundary conditions ($\square \in \{\text{free}, +\}$) is defined by a probability measure on $\Omega^{\square}_{\mathcal{G}}$ given by

$$\mathbf{P}^{\square}_{\mathcal{G},\beta}(\sigma) = \frac{1}{\mathcal{Z}^{\square}_{\mathcal{G},\beta}} \prod_{uv\in E} \exp\big(\beta J_{uv}\sigma_u\sigma_v\big), \qquad \sigma \in \Omega^{\square},$$

where

$$\mathcal{Z}^{\square}_{\mathcal{G},\beta} = \sum_{\sigma\in\Omega^{\square}} \prod_{uv\in E} \exp\big(\beta J_{uv}\sigma_u\sigma_v\big)$$

is the partition function. Recall the definition of the free energy density:

$$f_{\mathcal{G}}^{\square}(\beta) = -\frac{1}{\beta|V|} \ln \mathcal{Z}_{\mathcal{G},\beta}^{\square}.$$

It is clear that the free energy density is an analytic function of the inverse temperature $\beta \in (0, \infty)$ for every finite subtiling $\mathcal{G}$. However, when $\mathcal{G}$ approaches Γ, or more generally, some infinite subgraph of Γ (this is called taking a *thermodynamic limit*), the limiting function can have a *critical point*, i.e. a particular value of β where it is not analytic. The existence of such a point indicates that the system undergoes a phase transition when one varies β through the critical value.

Another approach, is to investigate the magnetic behavior of the system. To this end, one defines the *spin correlation functions*, i.e. the expectations of products of the *spin variables* taken with respect to the Ising probability measure. The simplest cases are the *one* and *two-point functions*

$$\langle \sigma_z \rangle_{\mathcal{G},\beta}^{\square} = \sum_{\sigma \in \Omega_{\mathcal{G}}^{\square}} \sigma_z \mathbf{P}_{\mathcal{G},\beta}^{\square}(\sigma), \quad \langle \sigma_z \sigma_w \rangle_{\mathcal{G},\beta}^{\square} = \sum_{\sigma \in \Omega_{\mathcal{G}}^{\square}} \sigma_z \sigma_w \mathbf{P}_{\mathcal{G},\beta}^{\square}(\sigma), \quad z, w \in V(\mathcal{G}).$$

Since the model is ferromagnetic, and due to the effect of positive boundary conditions, the corresponding one-point function $\langle \sigma_z \rangle_{\mathcal{G},\beta}^{+}$ is strictly positive for all finite subtilings $\mathcal{G}$ and for all β. In other words, in finite volume, the spins prefer the $+1$ state at all temperatures. However, when $\mathcal{G}$ approaches Γ, the boundary moves further and further away and, at temperatures high enough, its influence on a particular spin vanishes. As a result, the limiting one-point function equals zero and the spin equally likely occupies the $+1$ and -1 state. On the other hand, this does not happen at low temperatures, i.e. if β is sufficiently large, then $\langle \sigma_z \rangle_{\mathcal{G},\beta}^{+}$ stays bounded away from zero uniformly in $\mathcal{G}$. This means that the effect of positive boundary conditions is carried through all length scales and there is *spontaneous magnetization*. In this approach, the critical point is the value of β, which separates the regions with and without spontaneous magnetization. In some cases, it is more convenient to investigate the behavior of the two-point functions. Here, one also discerns two different non-critical cases: either the system is *disordered*, i.e. the thermodynamic limits of the two point functions decay exponentially fast to zero with the graph distance between z and w going to infinity, or the system is *ordered*, which means that the limiting two-point functions stay bounded away from zero uniformly in z and w. For periodic Ising models, the critical point defined as the value of β which separates these two regimes is the same as the critical point defined via spontaneous magnetization (see Theorem 1 in [1] and the references therein). In particular, the system exhibits long-range ferromagnetic order if and only if there is spontaneous magnetization.

Throughout this chapter, we will make a natural assumption on the coupling constants, namely we will require that there exist numbers m and M, such that for all $e \in E_{\Gamma}$,

$$0 < m \leq J_e \leq M < \infty. \tag{4.1}$$

Note that property (4.1) together with the conditions we imposed on Γ and Γ^* is enough for the existence of a phase transition in terms of spontaneous magnetization and the behavior of the two-point functions. This is a consequence of the classical arguments of Peierls [48] and Fisher [26]. In this chapter, we will only focus on the phase transition in the analytic behavior of the free energy density limits, but our results for the Kac–Ward operator can be also used in the setting of the magnetic phase transition (see Section 4.1.3).

4.1.1 The main result

Let $\vec{E}$ be the set of directed edges of $\mathcal{G}=(V,E)$ which are the ordered pairs of vertices. For a directed edge $\vec{e}=(u,v)$, we define its *reversion* by $-\vec{e}=(v,u)$ and we obtain the undirected version by dropping the arrow from the notation, i.e. $e=\{u,v\}$. If v is a vertex, then we write $\text{Out}(v)=\{(v',u')\in\vec{E}: v'=v\}$ for the set of edges emanating from v.

Let $\vec{x}$ and x be vectors of nonzero complex weights on the directed and undirected edges of $\mathcal{G}$ respectively. We call $\vec{x}$ *(Kac–Ward) contractive* on $\mathcal{G}$ if

$$\sum_{\vec{e}\in\text{Out}(v)} \arctan|\vec{x}_{\vec{e}}|^2 \leq \frac{\pi}{2} \qquad \text{for all } v\in V, \tag{4.2}$$

and we say that x *factorizes* to $\vec{x}$ if

$$x_e = \vec{x}_{\vec{e}}\vec{x}_{-\vec{e}} \qquad \text{for all } \vec{e}\in\vec{E}. \tag{4.3}$$

For the origin of condition (4.2), see Corollary 4.8.

In the context of the Ising model, two particular vectors of edge weights will be important, namely the so called *high* and *low-temperature* weights given by

$$\tanh\beta J = \big(\tanh\beta J_e\big)_{e\in E_\Gamma} \quad \text{and} \quad \exp(-2\beta J) = \big(\exp(-2\beta J_e)\big)_{e^*\in E^*_\Gamma}.$$

Definition 4.1. *We say that the coupling constants satisfy the* high-temperature *condition if* $\tanh J$ *factorizes to a contractive vector of weights on* Γ*, and they satisfy the* low-temperature *condition if* $\exp(-2J)$ *factorizes to a contractive vector of weights on* Γ^*.

Let

$$\Upsilon_\Box = \big\{ f^\Box_{\mathcal{G}} : \mathcal{G} \text{ is a finite subtiling of } \Gamma \big\}$$

be the family of all free energy densities with $\Box$ boundary conditions, and let $\overline{\Upsilon}_\Box$ be its closure in the topology of pointwise convergence on $(0,\infty)$. Note that $\overline{\Upsilon}_\Box$ contains all thermodynamic limits and can also contain other types of accumulation points of $\Upsilon_\Box$. Using the definition of $\mathcal{Z}^\Box_{\mathcal{G}}$, it is not difficult to prove that, under condition (4.1), $\Upsilon_\Box$ is uniformly bounded and equicontinuous on compact subsets of $(0,\infty)$. In particular, all

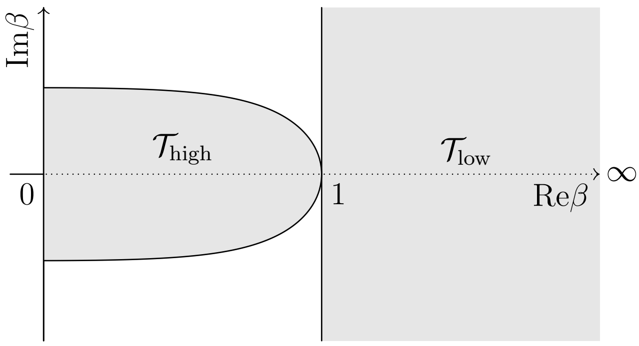

Figure 4.1: The high- and low-temperature regimes.

sequences in $\Upsilon_\square$ which converge pointwise, converge uniformly on compact sets, and therefore all functions in $\overline{\Upsilon}_\square$ are continuous on $(0, \infty)$. However, this is not enough to conclude analyticity of the limiting functions, and indeed, critical points do arise.

In this chapter, we will show that, if the coupling constants satisfy the high-temperature condition, then all functions in Υ_{free} can be extended analytically to a complex domain

$$\mathcal{T}_{\text{high}} = \Big\{\beta : \ 0 < \text{Re}\beta < 1, \ 2M|\text{Im}\beta| < \frac{\pi}{2}, \ \frac{\cosh(2m\text{Re}\beta)}{\cosh(2m)\cos(2M\text{Im}\beta)} < 1\Big\}$$

which we call the *high-temperature regime*. Note that $(0, 1) \subset \mathcal{T}_{\text{high}}$. Similarly we prove that, if the coupling constants satisfy the low-temperature condition, then all functions in Υ_+ can be extended to analytic functions on

$$\mathcal{T}_{\text{low}} = \{\beta : 1 < \text{Re}\beta\}$$

which we call the *low-temperature regime*. Moreover, we show that $\Upsilon_\square$ is uniformly bounded on compact subsets of the corresponding regimes.

For complex analytic functions, this is enough to conclude that all pointwise limits are also complex analytic. More precisely, let D be a complex domain and let $E \subset D$ have an accumulation point in D. The Vitali-Porter theorem (see [49, §2.4]) states that if a sequence of holomorphic functions defined on D converges pointwise on E, and is uniformly bounded on compact subsets of D, then it converges uniformly on compact subsets of D and the limiting function is holomorphic. In our context, the role of the domain D is played by the high- and low-temperature regimes, and E is the intersection of the given regime with the positive real numbers.

In other words, under the high- and low-temperature conditions on the coupling constants, the high- and low-temperature regimes are free of phase transition in terms of analyticity of the thermodynamic limits of the free energy density. This is summarized in the following theorem:

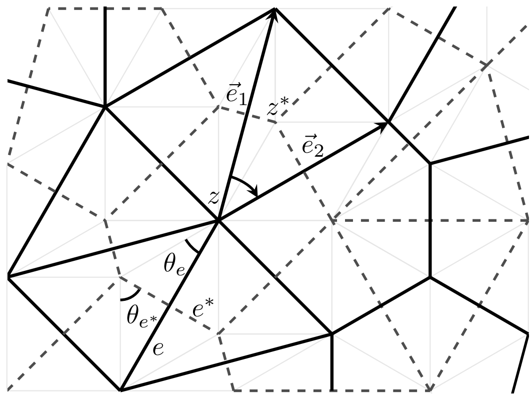

Figure 4.2: Local geometry of an isoradial graph and its dual. The underlying rhombic lattice is drawn in pale lines. The directed arc marks the turning angle $\angle(\vec{e}_1, \vec{e}_2)$.

Theorem 4.2. *If the coupling constants satisfy*

(i) the high-temperature condition, then all functions in $\Upsilon_{\rm free}$ extend analytically to $\mathcal{T}_{high}$, and $\Upsilon_{\rm free}$ is uniformly bounded on compact subsets of $\mathcal{T}_{high}$. As a consequence, all functions in $\overline{\Upsilon}_{\rm free}$ are analytic on $\mathcal{T}_{high}$, and in particular on $(0,1)$.

(ii) the low-temperature condition, then all functions in Υ_+ extend analytically to $\mathcal{T}_{low}$, and Υ_+ is uniformly bounded on compact subsets of $\mathcal{T}_{low}$. As a consequence, all functions in $\overline{\Upsilon}_+$ are analytic on $\mathcal{T}_{low}$, and in particular on $(1,\infty)$.

The proof of this theorem is provided in Section 4.3. Its main ingredients are the Kac–Ward formula for the partition function of the Ising model (see Theorem 4.4) and the bound on the spectral radius of the the Kac–Ward transition matrix given in Theorem 4.12.

In most of the applications, the role of boundary conditions is immaterial for the thermodynamic limit of the free energy density. Indeed, it is not hard to prove that whenever $|\partial \mathcal{G}|/|V|$ is small, then for $\beta \in (0,\infty)$, $f^{\rm free}_{\mathcal{G}}(\beta)$ and $f^+_{\mathcal{G}}(\beta)$ are close to each other (and also to any other free energy density function defined for other types of boundary conditions on $\mathcal{G}$). Hence, limits of the free energy density taken along sequences, where the above ratio approaches zero, are the same for all boundary conditions.

4.1.2 The isoradial case

Assume that Γ is an isoradial graph, i.e. all its faces can be inscribed in circles with a common radius, and all the circumcenters lie within the corresponding faces. An equivalent characterization says that Γ and Γ^* can be simultaneously embedded in the plane in such a way, that each pair of mutually dual edges forms diagonals of a rhombus. The roles of Γ and Γ^* are therefore symmetric and the dual graph is also isoradial. The simplest cases of isoradial graphs are the regular lattices: the square, triangular and hexagonal lattice.

One assigns to each edge e the interior angle θ_e that e creates with any side of the associated rhombus (see Figure 4.2). Note that $\theta_e + \theta_{e^*} = \pi/2$. There is a particular geometric choice of the coupling constants given by

$$\tanh J_e = \tan(\theta_e/2), \quad \text{or equivalently,} \quad \exp(-2J_e) = \tan(\theta_{e^*}/2). \tag{4.4}$$

These coupling constants were first considered by Baxter [4]. We will refer to them as the *self-dual Z-invariant* coupling constants since these are the only coupling constants that make the Ising model invariant under the star-triangle transformation, and also satisfy the above generalized Kramers-Wannier self-duality (4.4). For more details on their origin, see [3, 8].

Observe that in this setting, condition (4.1) is equivalent to the existence of constants k and K, such that for all $e \in E_\Gamma$,

$$0 < k \leq \theta_e \leq K < \pi. \tag{4.5}$$

This means that the associated rhombi have a positive minimal area, and also gives a uniform bound on the maximal degree of Γ and Γ^*.

The next corollary states that, for the Ising model defined by the above coupling constants, the only possible point of phase transition in the analytic behavior of the free energy density is $\beta = 1$.

Corollary 4.3. *Let Γ be an isoradial graph satisfying condition* (4.5)*. Consider Ising models defined by the self-dual Z-invariant coupling constants on finite subtilings of* Γ*. Then, all functions in $\overline{\Upsilon}_{\text{free}}$ are analytic on* $(0,1)$*, and all functions in $\overline{\Upsilon}_+$ are analytic on* $(1,\infty)$.

Proof. By (4.4) and the fact that the angles θ sum up to π around each vertex of Γ and Γ^*, the self-dual Z-invariant coupling constants simultaneously satisfy the high- and low-temperature condition. Indeed, the contractive weight vectors on the directed edges are given by $\vec{x}_{\vec{e}} = \sqrt{\tan(\theta_e/2)}$. The claim follows therefore from Theorem 4.2. $\square$

Note, that in this case, the inequalities in (4.2) become equalities.

4.1.3 Implications for the magnetic phase transition

In Chapter 2, the Kac–Ward operator and the signed weights it induces on the closed non-backtracking walks in a graph were used to rederive the critical temperature of the homogeneous Ising model on the square lattice. It was done both in terms of analyticity of the free energy density limit and the change in behavior of the one and two-point functions. The methods used there to analyze the correlation functions work also for general planar graphs under some slight regularity constraints. To be more precise, the proof of Theorem 2.5 which gives the existence of spontaneous magnetization, uses the fact that appropriate Kac–Ward transition matrices have spectral radius smaller than one and that the dual graph (which is Γ^* in this chapter) has subexponential growth of volume, i.e. the volume of balls in graph distance grows subexponentially with the radius. This condition is, for instance, satisfied by all isoradial graphs where (4.5) holds true. Moreover, Theorem 2.7 and Corollary 2.8, which yield exponential decay of the two-point functions, use the fact that the operator norm of appropriate Kac–Ward matrices is smaller than one.

The bounds that are stated in Section 4.2 allow to generalize the above results to arbitrary planar graphs, i.e. together with the methods from Chapter 2, they provide regions of parameters J and β where there is spontaneous magnetization or exponential decay of the two-point functions. These regions coincide with those in Theorem 4.2 (one can analytically extend the correlation functions to the high- and low-temperature regime), that is, if the coupling constants satisfy the low-temperature condition, then there is spontaneous magnetization on $\mathcal{T}_{\text{low}}$, and if they satisfy the high-temperature condition, then there is exponential decay of the two-point functions on $\mathcal{T}_{\text{high}}$. In particular, our bounds together with the methods developed in Chapter 2 establish that the self-dual Z-invariant weights are critical in the sense of magnetic phase transition.

We would also like to point out that the arguments, which are used in Chapter 2 to conclude analyticity of the free energy density limit, do not work for general graphs since they rely on periodicity of the square lattice. This is why, in this chapter, we go into details of this aspect of phase transition and we do not focus on the magnetic behavior of the model.

4.2 Results for the Kac–Ward operator

Let $\mathcal{G} = (V, E)$ be a finite simple graph embedded in the plane. We treat all edges as representative in the sense of Section 2.1.3. For a directed edge $\vec{e} = (u, v)$, we define its *tail* $t(\vec{e}) = u$ and *head* $h(\vec{e}) = v$. Recall that for $\vec{e}, \vec{g} \in \vec{E}$,

$$\angle(\vec{e}, \vec{g}) = \operatorname{Arg}\Big(\frac{h(\vec{g}) - t(\vec{g})}{h(\vec{e}) - t(\vec{e})}\Big) \in (-\pi, \pi] \tag{4.6}$$

is the turning angle from $\vec{e}$ to $\vec{g}$ (see Figure 4.2). The transition matrix for $\mathcal{G}$ and the weight vector x is given by

$$\Lambda_{\vec{e},\vec{g}}(x) = \begin{cases} x_e e^{\frac{i}{2}\angle(\vec{e},\vec{g})} & \text{if } h(\vec{e}) = t(\vec{g}) \text{ and } \vec{g} \neq -\vec{e}; \\ 0 & \text{otherwise,} \end{cases} \tag{4.7}$$

where $\vec{e}, \vec{g} \in \vec{E}$. Note that this matrix is equivalent to the matrix (2.13) if one does not consider additional edges. To each $\vec{e} \in \vec{E}$, we attach a copy of the complex numbers denoted by $\mathbb{C}_{\vec{e}}$ and we define a complex vector space

$$\mathcal{X} = \prod_{\vec{e}\in\vec{E}} \mathbb{C}_{\vec{e}}.$$

We identify $\Lambda(x)$ with the automorphism of $\mathcal{X}$ it defines via matrix multiplication. The *Kac–Ward operator* for $\mathcal{G}$ and the weight vector x is the automorphism of $\mathcal{X}$ given by

$$T(x) = \mathrm{Id} - \Lambda(x), \tag{4.8}$$

where Id is the identity on $\mathcal{X}$. As before, when necessary, we will use subscripts to express the fact that the above operators depend on the underlying graph $\mathcal{G}$.

If $\mathcal{G} = (V, E)$ is a finite subtiling of Γ, then we will denote by $\mathcal{G}^*$ the subgraph of Γ^* whose edge set consists of all the dual edges e^*, such that at least one of the endpoints of e belongs to $V \setminus \partial\mathcal{G}$. One can see that $\mathcal{G}^*$ is a subtiling of Γ^* whose defining set of dual faces is given by the vertices from $V \setminus \partial\mathcal{G}$. We will call it the *dual subtiling* of $\mathcal{G}$. Note that for subtilings containing holes, this definition does not agree with the definition of $\mathcal{G}^*$ from Chapter 3 (for the same reasons why the two definitions of $\partial\mathcal{G}$ did not agree). However, for both definitions it is true that the partition function of the Ising model on $\mathcal{G}$ with positive boundary conditions is given by the generating function of even subgraphs of $\mathcal{G}^*$.

The Kac–Ward formula expresses the square of an even subgraph generating function as the determinant of a Kac–Ward matrix with an appropriate edge weight vector. The combined result of the high- and low-temperature expansion together with the Kac–Ward formula is stated in the next theorem. For an account of the high- and low-temperature expansions, see Section 2.4.1.

Theorem 4.4. *Let $\mathcal{G} = (V, E)$ be a finite subtiling of Γ. For all choices of the coupling constants J and all β with $\mathrm{Re}\beta > 0$,*

$$\text{(i)} \qquad \big(\mathcal{Z}^{\mathrm{free}}_{\mathcal{G},\beta}\big)^2 = 2^{2|V|}\Big(\prod_{e\in E}\cosh^2(\beta J_e)\Big)\det\big[T_{\mathcal{G}}(\tanh\beta J)\big],$$

$$\text{(ii)} \qquad \big(\mathcal{Z}^{+}_{\mathcal{G},\beta}\big)^2 = \exp\Big(2\beta\sum_{e\in E} J_e\Big)\det\big[T_{\mathcal{G}^*}(\exp(-2\beta J))\big].$$

Note that the condition $\mathrm{Re}\beta > 0$ is needed only for the weight vector $\tanh\beta J$ to be well defined.

The determinant of the Kac–Ward matrix is the characteristic polynomial of the transition matrix evaluated at one:

$$\det T = \det(\mathrm{Id} - \Lambda) = \prod_{k=1}^{2n}(1-\lambda_k),$$

where n is the number of edges of $\mathcal{G}$, and λ_k, $k \in \{1, 2, \ldots, 2n\}$, are the eigenvalues of Λ. Recall that we want to extend the free energy density functions to domains in the complex plane. The free energy density is given by the logarithm of the partition function, and the square of the partition function is proportional to the above product involving eigenvalues of the transition matrix. In this situation, it is natural to use the power series expansion of the logarithm around 1:

$$\ln(1-\lambda) = -\sum_{r=1}^{\infty} \lambda^r / r, \qquad |\lambda| < 1.$$

This series is convergent whenever λ stays within the unit disc, and hence we should require that the spectral radius of the transition matrix is bounded from above by 1. The next section is devoted to providing the necessary estimates.

4.2.1 Bounds on the spectral radius and operator norm

In this section, we will make use of transition matrices conjugated by diagonal matrices of a certain type: if x factorizes to $\vec{x}$, i.e. $\vec{x}_{\vec{e}}\vec{x}_{-\vec{e}} = x_e$ for all $\vec{e} \in \vec{E}$, then we define the *conjugated transition matrix* by

$$\Lambda(\vec{x}) = D^{-1}(\vec{x})\Lambda(x)D(\vec{x}),$$

where $D(\vec{x})$ is the diagonal matrix satisfying $D_{\vec{e},\vec{e}}(\vec{x}) = \vec{x}_{\vec{e}}$ for all $\vec{e} \in \vec{E}$. The resulting transition matrix takes the following form:

$$\Lambda_{\vec{e},\vec{g}}(\vec{x}) = \begin{cases} \vec{x}_{-\vec{e}}\vec{x}_{\vec{g}} e^{\frac{i}{2}\angle(\vec{e},\vec{g})} & \text{if } h(\vec{e}) = t(\vec{g}) \text{ and } \vec{g} \neq -\vec{e}; \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

This matrix is similar to the standard transition matrix, and in particular has the same spectrum. Moreover, it turns out that one can explicitly compute its operator norm.

To this end, let us make some additional observations. For a square matrix A, let $\|A\|$ be its operator norm induced by the Euclidean norm, and let $\rho(A)$ be its spectral radius. Note that there is a natural involutive automorphism P of $\mathcal{X}$ induced by the map $\vec{e} \mapsto -\vec{e}$, i.e. the automorphism which assigns to each complex number in $\mathbb{C}_{\vec{e}}$ the same complex number in $\mathbb{C}_{-\vec{e}}$. Fix $\vec{x}$ and let $A = P\Lambda(\vec{x})$. Observe that $\|A\| = \|\Lambda(\vec{x})\|$ since P is an isometry. Moreover, the operator norm of A depends only on the absolute values of $\vec{x}$. Indeed, if

$$B = D(\vec{u})AD(\vec{u}), \qquad \text{where} \quad \vec{u}_{\vec{e}} = |\vec{x}_{\vec{e}}|/\vec{x}_{\vec{e}},$$

then B is given by the matrix

$$B_{\vec{e},\vec{g}} = \begin{cases} |\vec{x}_{\vec{e}}\vec{x}_{\vec{g}}| e^{\frac{i}{2}\angle(-\vec{e},\vec{g})} & \text{if } t(\vec{e}) = t(\vec{g}) \text{ and } \vec{g} \neq \vec{e}; \\ 0 & \text{otherwise,} \end{cases} \tag{4.10}$$

and $\|B\| = \|A\|$ since $D(\vec{u})$ is an isometry.

Note that $\mathcal{X}$ can be decomposed as

$$\mathcal{X} = \prod_{v \in V} \mathcal{X}^v, \qquad \text{where} \quad \mathcal{X}^v = \prod_{\vec{e} \in \mathrm{Out}(v)} \mathbb{C}_{\vec{e}}.$$

One can see from (4.10) that B gives a nonzero transition weight only between two edges sharing the same tail v. In other words, B maps $\mathcal{X}^v$ to itself and therefore is block-diagonal, that is

$$B = \prod_{v \in V} B^v,$$

where $B^v : \mathcal{X}^v \to \mathcal{X}^v$ is the restriction of B to the space $\mathcal{X}^v$. Moreover, the angles satisfy

$$\angle(-\vec{e}, \vec{g}) = -\angle(-\vec{g}, \vec{e}) \qquad \text{for} \quad \vec{e} \neq \vec{g}, \tag{4.11}$$

and hence B is Hermitian, i.e. $B_{\vec{e},\vec{g}} = \overline{B_{\vec{g},\vec{e}}}$. Combining these two properties and the fact that the operator norm of a Hermitian matrix is given by its spectral radius, we arrive at the identity:

$$\|B\| = \rho(B) = \max_{v \in V} \rho(B^v). \tag{4.12}$$

It turns out that the characteristic polynomial of B^v is easily expressible in terms of the weight vector $\vec{x}$:

Lemma 4.5. *For any real t and any vertex v,*

$$\det(t\mathrm{Id} - B^v) = \mathrm{Re}\Big(\prod_{\vec{e} \in \mathrm{Out}(v)} (t + i|\vec{x}_{\vec{e}}|^2) \Big),$$

where Id *is the identity on $\mathcal{X}^v$.*

Proof. The proof is by induction on the degree of v. One can easily check that the statement is true for all vertices of degree one or two. Now suppose that it is true for all vertices of degree at most $n \geq 2$. Let v be a vertex of degree $n+1$ and let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{n+1}$ be a counterclockwise ordering of the edges of Out(v). Consider the matrix $S = t\mathrm{Id} - B^v$ with columns and rows ordered accordingly. Note that for all $\vec{g} \in$ Out(v) different from $\vec{e}_1$ and $\vec{e}_2$,

$$\angle(\vec{g}, \vec{e}_1) + \angle(\vec{e}_1, \vec{e}_2) + \angle(\vec{e}_2, \vec{g}) = 0 \ (\mathrm{mod}\ 2\pi).$$

Also observe that, for geometric reasons, at least two of the above angles are positive. Combining this together with the fact that Arg$(w) =$ Arg$(-w) \pm \pi$ for any complex w, and that the angles are between $-\pi$ and π, yields

$$\angle(-\vec{e}_1, \vec{g}) = \angle(-\vec{e}_2, \vec{g}) + \angle(-\vec{e}_1, \vec{e}_2) + \pi. \tag{4.13}$$

Using this identity, we will perform row and column operations on S which will simplify the computation of the determinant. We first subtract from the first row of S, the second row multiplied by $ie^{\frac{i}{2}\angle(-\vec{e}_1,\vec{e}_2)}|\vec{x}_{\vec{e}_1}|/|\vec{x}_{\vec{e}_2}|$. Then, we subtract from the first column the second one multiplied by $-ie^{-\frac{i}{2}\angle(-\vec{e}_1,\vec{e}_2)}|\vec{x}_{\vec{e}_1}|/|\vec{x}_{\vec{e}_2}|$. The resulting matrix has the same determinant as S. By the definition of B^v, (4.11) and (4.13),

$$\det S = \det \begin{pmatrix} a & b & 0 & 0 & \cdots \\ \bar{b} & t & -B^v_{\vec{e}_2,\vec{e}_3} & -B^v_{\vec{e}_2,\vec{e}_4} & \cdots \\ 0 & -\overline{B^v_{\vec{e}_2,\vec{e}_3}} & t & -B^v_{\vec{e}_3,\vec{e}_4} & \cdots \\ 0 & -\overline{B^v_{\vec{e}_2,\vec{e}_4}} & -\overline{B^v_{\vec{e}_3,\vec{e}_4}} & t & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $a = t\big(1+|\vec{x}_{\vec{e}_1}|^2/|\vec{x}_{\vec{e}_2}|^2\big)$ and $b = -e^{\frac{i}{2}\angle(-\vec{e}_1,\vec{e}_2)}\big(it|\vec{x}_{\vec{e}_1}|/|\vec{x}_{\vec{e}_2}| + |\vec{x}_{\vec{e}_1}\vec{x}_{\vec{e}_2}|\big)$.

Let S_1 be the matrix resulting from removing from S the first column and the first row, and let S_2 be the matrix, where the first two rows and the first two columns of S are removed. By the induction hypothesis, $\det S_1 = \mathrm{Re}\big((t+i|\vec{x}_{\vec{e}_2}|^2)\vartheta\big)$ and $S_2 = \mathrm{Re}\vartheta$, where $\vartheta = \prod_{\vec{g}\in\mathrm{Out}(v)\setminus\{\vec{e}_1,\vec{e}_2\}}(t+i|\vec{x}_{\vec{g}}|^2)$. Expanding the determinant, we get

$$\begin{aligned} \det S &= a\det S_1 - b\bar{b}\det S_2 \\ &= t\big(1+|\vec{x}_{\vec{e}_1}|^2/|\vec{x}_{\vec{e}_2}|^2\big)\mathrm{Re}\big((t+i|\vec{x}_{\vec{e}_2}|^2)\vartheta\big) \\ &\quad - \big(|\vec{x}_{\vec{e}_1}|^2|\vec{x}_{\vec{e}_2}|^2 + t^2|\vec{x}_{\vec{e}_1}|^2/|\vec{x}_{\vec{e}_2}|^2\big)\mathrm{Re}\vartheta \\ &= \mathrm{Re}\big((t+i|\vec{x}_{\vec{e}_1}|^2)(t+i|\vec{x}_{\vec{e}_2}|^2)\vartheta\big). \end{aligned}$$

The last equality follows since both sides are real linear in ϑ, and one can check that it holds true for $\vartheta = 1, i$. ☐

For $v \in V$, we define $\xi^v(\vec{x})$ to be the unique solution in s of the equation

$$\sum_{\vec{e}\in\mathrm{Out}(v)} \arctan\big(|\vec{x}_{\vec{e}}|^2/s\big) = \frac{\pi}{2}. \tag{4.14}$$

As a corollary we obtain the following result:

Corollary 4.6. $\rho(B^v) = \xi^v(\vec{x})$.

Proof. Since B^v is Hermitian, it has a real spectrum. By Lemma 4.5, the characteristic polynomial of B^v at a nonzero real number t is given by

$$t^{|\mathrm{Out}(v)|}\Big(\prod_{\vec{e}\in\mathrm{Out}(v)}\cos\big(\arctan(|\vec{x}_{\vec{e}}|^2/t)\big)\Big)^{-1}\cos\Big(\sum_{\vec{e}\in\mathrm{Out}(v)}\arctan(|\vec{x}_{\vec{e}}|^2/t)\Big).$$

This expression vanishes only when the last cosine term is zero. The largest in modulus values of t for which this happens are equal to $\pm\xi^v(\vec{x})$. ☐

We can now compute the operator norm of the conjugated transition matrix $\Lambda(\vec{x})$. The following result is the main tool in our considerations:

Lemma 4.7.

$$\|\Lambda(\vec{x})\| = \max_{v \in V} \xi^v(\vec{x}).$$

Proof. It follows from the fact that $\|\Lambda(\vec{x})\| = \|B\|$, identity (4.12), and Corollary 4.6.□

Note that the operator norm depends only on the absolute values of $\vec{x}$. One can rephrase this result as follows:

Corollary 4.8. $\|\Lambda(\vec{x})\| \leq s$ *if and only if*

$$\sum_{\vec{e} \in \mathrm{Out}(v)} \arctan\left(|\vec{x}_{\vec{e}}|^2/s\right) \leq \frac{\pi}{2} \qquad \textit{for all } v \in V.$$

We say that an operator is a *contraction* if its operator norm is smaller or equal to 1, and hence the name of condition (4.2). Since the operator norm bounds the spectral radius from above, we obtain the following corollary:

Corollary 4.9. *If x factorizes to $\vec{x}$, then*

$$\rho(\Lambda(x)) \leq \max_{v \in V} \xi^v(\vec{x}).$$

This inequality is preserved when one takes the infimum over all factorizations of the weight vector x. One can check that the spectral radius of the transition matrix depends not only on the moduli but also on the complex arguments of x. Since the above bound depends only on the absolute values, it is in general not sharp. Nonetheless, it is optimal for the self-dual Z-invariant Ising model on isoradial graphs.

Remark 4.10. *Note that finiteness of $\mathcal{G}$ was not important in our computations. Since the transition matrix is defined locally for each vertex, we only used the fact that all vertices have finite degree. Hence, one can consider transition matrices and Kac–Ward operators on infinite graphs as automorphisms of the Hilbert space ℓ^2 on the directed edges of $\mathcal{G}$. The results from this section translate directly to this setting by interchanging all maxima with suprema. This is used in Chapter 5 to analyse infinitely dimensional Kac–Ward operators.*

Remark 4.11. *Note that the results from this section, when applied to the square lattice, recover the bounds from Theorem 2.11. Moreover, the general idea behind the proofs of these results is the same as the one from the proof of Theorem 2.11.*

4.2.2 High- and low-temperature spectral radii

We will now use the bounds from the previous section in a more concrete setting of the high- and low-temperature weight vectors. We define

$$R(\beta) = \sup_{\mathcal{G}} \rho_{\mathcal{G}}(\tanh \beta J) \quad \text{and} \quad R^*(\beta) = \sup_{\mathcal{G}} \rho_{\mathcal{G}^*}(\exp(-2\beta J)),$$

where the suprema are taken over all finite subtilings of Γ, and where $\rho_{\mathcal{G}}(x)$ is the spectral radius of the transition matrix $\Lambda_{\mathcal{G}}(x)$. The reason for our particular choice of the high- and low-temperatures regimes in the statement of Theorem 4.2 is the following result:

Theorem 4.12. *If the coupling constants satisfy*

(i) the high-temperature condition, then $\sup_{\beta \in K} R(\beta) < 1$ *for any compact set* $K \subset \mathcal{T}_{high}$.

(ii) the low-temperature condition, then $\sup_{\beta \in K} R^*(\beta) < 1$ *for any compact set* $K \subset \mathcal{T}_{low}$.

Proof. We will prove part (i). Fix a compact set $K \subset \mathcal{T}_{\text{high}}$ and let

$$L(\beta) = \sup_{j \in [m,M]} \frac{|\tanh \beta j|}{\tanh j} = \sup_{j \in [m,M]} \frac{\cosh j}{\sinh j} \sqrt{\frac{\cosh(2j\mathrm{Re}\beta) - \cos(2j\mathrm{Im}\beta)}{\cosh(2j\mathrm{Re}\beta) + \cos(2j\mathrm{Im}\beta)}}.$$

By compactness of $[m, M]$ and the fact that the hyperbolic tangent does not vanish and is continuous in the right half-plane, L is a continuous function on $\{\beta : 0 < \mathrm{Re}\beta\}$. From a simple computation, it follows that $L(\beta) < 1$ if and only if

$$\cosh(2j\mathrm{Re}\beta)/\cosh 2j < \cos(2j\mathrm{Im}\beta) \qquad \text{for all } j \in [m, M].$$

The above inequality can hold only when $|\mathrm{Re}\beta| < 1$ and when the right hand side is positive. The latter is in particular true when $2M|\mathrm{Im}\beta| < \frac{\pi}{2}$. Under these assumptions, both sides of the inequality are decreasing functions of j. This means that the above condition is satisfied whenever $0 < \mathrm{Re}\beta < 1$, $2M|\mathrm{Im}\beta| < \frac{\pi}{2}$ and $\cosh(2m\mathrm{Re}\beta)/\cosh 2m < \cos(2M\mathrm{Im}\beta)$. Hence, by the definition of $\mathcal{T}_{\text{high}}$, we have that $\mathcal{T}_{\text{high}} \subset \{\beta : L(\beta) < 1\}$ and thus, by continuity of L,

$$s := \sup_{\beta \in K} L(\beta) < 1.$$

From the definition of L, it follows that

$$|\tanh \beta J_e|/\tanh J_e \le s \qquad \text{for all } e \in E_\Gamma \text{ and } \beta \in K.$$

We assume that the coupling constants J satisfy the high-temperature condition, which means that the weight vector $\tanh J$ factorizes to a contractive weight vector $\vec{x}$. It follows that for $\beta \in K$, $\tanh \beta J$ factorizes to a weight vector $\vec{x}(\beta)$ satisfying $|\vec{x}_{\vec{e}}(\beta)| = \sqrt{|\tanh \beta J_e|/\tanh J_e} \cdot |\vec{x}_{\vec{e}}|$, and hence $|\vec{x}_{\vec{e}}(\beta)|^2/s \le |\vec{x}_{\vec{e}}|^2$ for all $\vec{e} \in \vec{E}_\Gamma$. Since $\arctan$ is increasing and $\vec{x}$ is contractive, we have by Corollary 4.8 that $\|\Lambda_{\mathcal{G}}(\vec{x}(\beta))\| \le s$ for all subtilings $\mathcal{G}$ and all $\beta \in K$, where $\Lambda_{\mathcal{G}}$ is the transition matrix defined for the graph $\mathcal{G}$. The claim follows because the spectral radius is bounded from above by the operator norm, and $\Lambda_{\mathcal{G}}(\vec{x}(\beta))$ has the same spectral radius as $\Lambda_{\mathcal{G}}(\tanh \beta J)$.

Part (ii) involves less computations and can be proved similarly after noticing that

$$\mathcal{T}_{\text{low}} = \Big\{ \beta : \sup_{j\in[m,M]} \frac{|\exp(-2\beta j)|}{\exp(-2j)} < 1 \Big\}. \qquad \square$$

In the light of Thoerem 4.4 and the remarks which follow it, we are now in a position to prove our main result.

4.3 Proof of the main result

Proof. We will prove part (i). Suppose that the coupling constants satisfy the high-temperature condition and fix a compact set $K \subset \mathcal{T}_{\text{high}}$. We have to show that the functions $f^{\text{free}}_{\mathcal{G}}$ extend analytically to $\mathcal{T}_{\text{high}}$ and are uniformly bounded on K.

First of all, since zero is not in $\mathcal{T}_{\text{high}}$, the factor $1/\beta$ is analytic on $\mathcal{T}_{\text{high}}$ and uniformly bounded on K. Thus, it is enough to consider functions of the form $\ln \mathcal{Z}^{\text{free}}_{\mathcal{G},\beta}/|V|$ where $\mathcal{G} = (V, E)$. We will use the formula from part (i) of Theorem 4.4. The logarithm of the partition function can therefore be expressed as a sum of three different terms. The first one is the constant $|V|\ln 2$, which equals $\ln 2$ after rescaling by the number of vertices.

To talk about the second term, which comes from the product of hyperbolic cosines, one has to argue that there is a continuous branch of $\ln(\cosh \beta J_e)$ on $\mathcal{T}_{\text{high}}$. Indeed, one can take the principal value of the logarithm since

$$\text{Re}(\cosh \beta J_e) = \cosh(J_e \text{Re}\beta)\cos(J_e \text{Im}\beta) > 0 \quad \text{on } \mathcal{T}_{\text{high}}.$$

Analyticity of this term follows since $\cosh \beta J_e$ is analytic. Furthermore, we have

$$\begin{aligned} \Big| \ln\Big(\prod_{e\in E} \cosh \beta J_e \Big) \Big| &\leq \sum_{e\in E} \big| \ln(\cosh \beta J_e) \big| \\ &\leq \sum_{e\in E} \Big(\big| \ln|\cosh \beta J_e| \big| + |\text{Arg}(\cosh \beta J_e)| \Big) \\ &\leq |E| \Big(\sup_{j\in[m,M]} \big| \ln|\cosh \beta j| \big| + \pi/2 \Big). \end{aligned}$$

Since the hyperbolic cosine does not vanish in the right half-plane and $[m, M]$ is compact, the above supremum is a continuous function of β on $\mathcal{T}_{\text{high}}$, and therefore is bounded on K. The number of edges is bounded by the number of vertices times the maximal degree of Γ, and thus, after rescaling by the volume, this term is uniformly bounded in $\mathcal{G}$.

The last term is given by the logarithm of the determinant of the Kac–Ward operator. Let λ_k, $k \in \{1, 2, \ldots, 2n\}$, $n = |E|$, be the eigenvalues of the transition matrix $\Lambda(\tanh \beta J)$ defined for $\mathcal{G}$. By Theorem 4.12, we know that their moduli are bounded from above by some constant $s < 1$ (uniformly in $\mathcal{G}$ and $\beta \in K$). One can therefore define the logarithm by its power series around 1, i.e.

$$\begin{aligned}\ln\det\big[\mathrm{Id}-\Lambda(\tanh\beta J)\big] &= \ln\prod_{k=1}^{2n}(1-\lambda_k) = \sum_{k=1}^{2n}\ln(1-\lambda_k)\\ &= -\sum_{k=1}^{2n}\sum_{r=1}^{\infty}\lambda_k^r/r = -\sum_{r=1}^{\infty}\sum_{k=1}^{2n}\lambda_k^r/r\\ &= -\sum_{r=1}^{\infty}\mathrm{tr}[\Lambda^r(\tanh\beta J)]/r.\end{aligned}$$

It is clear that $\mathrm{tr}[\Lambda^r(\tanh\beta J)]$ is an analytic function of β. Moreover, one has that $|\mathrm{tr}[\Lambda^r(\tanh\beta J)]| \leq 2|E|s^r$ for all r, and therefore the above series converges uniformly on K. It follows that the series defines a holomorphic function on $\mathcal{T}_{\text{high}}$. Again, after rescaling by the number of vertices, it becomes uniformly bounded in $\mathcal{G}$. This completes the proof of the first part of the theorem.

The proof of part (ii) uses the second formula from Theorem 4.4 and proceeds in a similar manner. $\square$